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GLOBAL REGULARIZATION OF THE ORBITS OF RELATIVE MOTION IN A VERSION OF THE MANY-BODY PROBLEM†

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A globally regularized system of differential equations of the motion of a point mass with respect to the main centre of attraction is set up within the framework of the many-body problem, taking into account perturbations of gravitating points performing circular motions about the centre. Formulae of the first approximation of the solution of the system are derived. An asymptotic expansion of the general solution of the equivalent canonical equations describing the motion of a point mass is proposed using the zeroth (Kepler) and first approximations. The region of expansion of the solution in series in powers of the initial values of the phase coordinates and a parameter is determined. © 1997 Elsevier Science Ltd. All rights reserved.

Existing asymptotic representations of the solution of the many-body problem contain initial terms of the expansions in powers of regular time [1], of the sequence of Picard approximations in certain regions of phase space [2, 3], and the first approximation in the neighbourhood of the generating solution [4].

The method of constructing a solution of a version of the many-body problem proposed below can be regarded as a supplement to the above papers in the sense that it provides an accurate representation of the regularized orbits of the relative motion of a point mass.

Consider a mechanical system consisting of n points $M_1, \ldots, M_n$ with masses $m_1, \ldots, m_n$. Suppose $\mathbf{r}_j$ is the radius vector of the point M_j in a certain inertial orthogonal system of coordinates. Within the framework of the Newtonian n-body problem the equation of motion of the point M_2 with respect to the point M_1 has the form [5]

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = -\sum_{i=3}^{n} \frac{\mu_i}{d_i^3}\mathbf{d}_i + \frac{\mu_i}{R_i^3}\mathbf{R}_i \tag{1}$$

$$\mu = \gamma(m_1 + m_2),\ \mu_i = \gamma m_i,\ \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,\ \mathbf{R}_i = \mathbf{r}_i - \mathbf{r}_1,\ \mathbf{d}_i = \mathbf{r} - \mathbf{R}_i$$

(γ is the gravitational constant).

In order to convert the first part of the equation we will introduce the following variables

$$q_i = 2R_i^{-2}(\mathbf{R}_i \cdot \mathbf{r} - r^2 / 2), \quad \varphi = (1 - q_i)^{-3/2} - 1 \tag{2}$$

The equation

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = \sum_{i=3}^{n} \frac{\mu_i}{R_i^3}[\varphi_i \mathbf{R}_i - (1+\varphi_i)\mathbf{r}] = \mathbf{f} \tag{3}$$

is then equivalent to Eq. (1) by virtue of relations (2).

We will further assume that the point M_i $(i = 3, \ldots, n)$ moves with respect to M_1 in a circular orbit of radius R_i, given by the equation [6]

$$\mathbf{R}_i = R_i \cos\left(E_{i0} + \sqrt{\frac{\mu}{R_i^3}}t\right)\mathbf{A}_i + R_i \sin\left(E_{i0} + \sqrt{\frac{\mu}{R_i^3}}t\right)\mathbf{B}_i, \quad R_{i+1} > R_i$$

in which E_{i0} is the initial value of the eccentric anomaly E_i, while the vectors $\mathbf{A}_i$ and $\mathbf{B}_i$ are constants.

From Eq. (3), using the Sundman transformation $dt = r\,ds$ and the Kustaanheimo–Stiefel transformation

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$$r = \Lambda(\mathbf{u})\mathbf{u}, \quad r = u^2, \quad \mathbf{u} = (u_1, u_2, u_3, u_4)^T$$

$$\Lambda(\mathbf{u}) = \begin{Vmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{Vmatrix} \tag{4}$$

we obtain the regularized system of equations [6]

$$\mathbf{u}' = \mathbf{w}, \quad \mathbf{w}' = -\frac{h}{2}\mathbf{u} + \sum_{i=3}^{n} \frac{\mu_i u^2}{2R_i^3}[\varphi_i \Lambda^T \mathbf{R}_i - u^2(1+\varphi_i)\mathbf{u}] \tag{5}$$

in which $\mathbf{u}$ is the *KS*-position vector and the prime denotes differentiation with respect to s.

The above equations enable us to write equations for the variable φ_i, the total mechanical energy $(-h)$ of the point M_2 and the time t as follows:

$$\varphi_i' = \frac{3(1+\varphi_i)^{5/3}}{R_i^2}\left(\sqrt{\frac{\mu}{R_i^3}}u^2\left(\frac{\partial \mathbf{R}_i}{\partial E_i}, \ \mathbf{r}\right) + 2(\mathbf{R}_i, \ \Lambda\mathbf{u}') - 2(\mathbf{r}, \ \Lambda\mathbf{u}')\right)$$

$$h' = -2(\mathbf{u}', \ \Lambda^T\mathbf{f}), \quad t' = u^2 \tag{6}$$

Hence, we have compiled a globally regularized system of equations (5), (6) in the phase variables $\mathbf{u}, \mathbf{w}, \varphi_i, h, t$. We will assume that at the instant $t = 0$ $(s = 0)$ we are given the initial values of $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, which, by virtue of the transition regularization formulae, define the initial values $\mathbf{u}^{(0)}$, $\mathbf{w}^{(0)}$.

We will consider the following procedure for finding a solution of the Cauchy problem for system of equations (5), (6).

We will take as the initial zeroth approximation of the solution of the universal Kepler solution, denoted by the subscript c [6]

$$\begin{aligned} &\mathbf{u}_c = c_0(h_c s^2/2)\mathbf{u}^{(0)} + s c_1(h_c s^2/2)\mathbf{w}^{(0)} \\ &h_c = \mu/r_0 - \dot{\mathbf{r}}_0^2/2, \quad t_c = r_\pi s + \mu e s^3 C_3(2h_c s^2) \\ &\varphi_{ic} = (1-q_{ic})^{-3/2} - 1, \quad q_{ic} = 2R_i^{-2}(\mathbf{R}_i \cdot \mathbf{r}_c - r_c^2/2), \quad \mathbf{r}_c \neq \mathbf{R}_i \end{aligned} \tag{7}$$

Here r_π and e are the radius of the pericentre and the eccentricity of the Kepler orbit, respectively, while $c_j(\cdot)$ are the symbols of the Stumpff functions.

It should be noted that the function $t_c(s)$ is invertible with respect to the argument s.

Starting from the third equation of (7) we obtain the relation [7]

$$s = \frac{t_c}{r_\pi} + \sum_{n=1}^{\infty} \varepsilon^n Q_n\left(\frac{t_c}{r_\pi}\right), \quad \varepsilon = \frac{\mu e}{r_\pi}$$

The quantities Q_n are found from the recurrence formulae

$$Q_{n+1} = \frac{1}{n}\sum_{k=1}^{n} \frac{\partial Q_{n+1-k}}{\partial \theta} Q_k, \quad \theta = \frac{t_c}{r_\pi}$$

with the initial value

$$Q_1(\theta) = -\theta^3 c_3(2h_c\theta^2)$$

We introduce the parameter $\lambda = \max_i \mu_i$, for which $\mu_i = \lambda_i\lambda$, and we write the first approximation of the solution of system (5), (6) with respect to λ as follows:

$$u^{(1)} = u_c + \lambda\Delta u, \quad t^{(1)} = t_c + \lambda\Delta t, \quad h^{(1)} = h_c + \lambda\Delta h$$

From (5) we can write an equation for $\Delta\mathbf{u}$ in the form

$$(\Delta \mathbf{u})'' + \frac{h_c}{2}\Delta \mathbf{u} = -\frac{\Delta h}{2}\mathbf{u}_c + \sum_{i=3}^{n} \frac{\lambda_i u_c^2}{2R_i^3}[\varphi_{ic}\Lambda_c^T \mathbf{R}_i - u_c^2(1+\varphi_{ic})\mathbf{u}_c] = \{g_k(s)\} \tag{8}$$

$$\Delta h = -\frac{2}{\lambda}\int_0^s (\mathbf{u}_c', \ \Lambda_c^T \mathbf{f}_c)\, ds$$

in which $g_k(s)$ is an abbreviated form of writing the right-hand side of the kth equation.
Using the method of variation of constants, we can write the solution of Eq. (8) as

$$\Delta u_k = \int_0^s (s-\vartheta)c_1\left(\frac{h_c}{2}(s-\vartheta)^2\right) g_k(\vartheta)\, d\vartheta \tag{9}$$

Correspondingly we have

$$\Delta t = 2\int_0^s (\mathbf{u}_c, \ \Delta \mathbf{u})\, ds$$

To find the expansion of the solution it is best to consider the following Hamiltonian referred to regular time s [6]

$$\Gamma = \frac{1}{2}w^2 + \frac{1}{2}w_0 u^2 + \frac{1}{4}V(u_0, \ \mathbf{u})u^2 - \frac{\mu}{4} \tag{10}$$

in which the canonical variables u_k and w_k, by virtue of [6], are $u_0 = t/2$ and $w_0 = h/2$, while for values of the subscript $k = 1, 2, 3, 4$, the quantities u_k and w_k are the components of the vectors of parametric position and parametric velocity, respectively.

Provided that, in the interval of the time of motion considered $s \in I$ the gravitating points M_i ($i \geqslant i_0$) are external to the point M_2 in the sense that the inequality $r < R_{i_0}$ is satisfied, we can write the potential function $V_{i_0}(u_0, \mathbf{u})$ in the form [8]

$$V_{i_0} = -\sum_{i=i_0}^{n}\sum_{k=2}^{\infty} \frac{\mu_i r^k}{R_i^{k+1}} P_k(\cos H_i), \quad \cos H_i = \frac{\mathbf{r}\mathbf{R}_i}{rR_i} \tag{11}$$

If for $s \in I$, the point M_i ($i < i_0$) is internal with respect to M_2 in the sense that the inequality $r \geqslant R_{i_0-1}$ is satisfied, the potential function $\tilde{V}_{i_0}$ has the form [6]

$$\tilde{V}_{i_0} = -\sum_{i=3}^{i_0-1}\sum_{k=2}^{\infty} \frac{(\mu+\mu_i)m_k}{(m_1+m_i)r^{k+1}} R_i^k P_k(\cos H_i) \tag{12}$$

We will assume that the inequality $r < R$ is satisfied over the whole interval $0 \leqslant s \leqslant \alpha$ considered. The following expansion then holds

$$\frac{1}{r} = \frac{1}{R}\left(1 + \frac{R-r}{R} + \frac{(R-r)^2}{R^2} + \ldots\right) \tag{13}$$

Note that if $r < R_3$, $i_0 = 3$, we have $\tilde{V}_{i_0} = 0$, while if $r > R_n$, $i_0 = n+1$ we have $V_{i_0} = 0$. Hence, we obtain

$$V(u_0, \ \mathbf{u}) = V_{i_0} + \tilde{V}_{i_0}, \quad s \in I \tag{14}$$

We will compile the solution of the canonical system of equations with Hamiltonian (10)

$$\mathbf{U}' = \frac{\partial \Gamma}{\partial \mathbf{W}}, \quad \mathbf{W}' = -\frac{\partial \Gamma}{\partial \mathbf{U}}, \quad \mathbf{U} = (u_0, \ u_1, \ u_4), \quad \mathbf{W} = (w_0, \ w_1, \ \ldots w_4) \tag{15}$$

using the approximations obtained in the form of asymptotic expansions

$$\mathbf{U} = \mathbf{U}_c + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbf{q}^{(n)}(\mathbf{U}_c, \mathbf{W}_c), \quad \mathbf{W} = \mathbf{W}_c + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbf{p}^{(n)}(\mathbf{U}_c, \mathbf{W}_c) \tag{16}$$

To determine the vector function $q^{(n)}, p^{(n)}$ we will use the Deprit–Kamel algorithm [9]

$$q^{(n)} = \frac{\partial \Gamma_n}{\partial \mathbf{W}_c} + \sum_{j=1}^{n-1} C_{n-1}^{j} \mathbf{q}_{j,n-j}, \quad \Gamma_n = \left.\frac{\partial^{(n-1)} \Gamma}{\partial \lambda^{(n-1)}}\right|_{\lambda=0}$$

$$\mathbf{q}_{ji} = B_j \mathbf{q}^{(i)} - \sum_{m=1}^{j-1} C_{j-1}^{m-1} B_m \mathbf{q}_{j-m,i}, \quad \mathbf{q}_{11} = B_1 \mathbf{q}^{(1)}, \quad \mathbf{q}^{(1)} = \left(\frac{\Delta t}{2}, \ \Delta \mathbf{u}^T\right)$$

The result of applying the operator B_i to the scalar function ψ leads to the calculation of the Poisson bracket $B_i\psi = \{\psi, \Gamma_i\}$.

When solving the Cauchy problem using (16) the function $V(u_0, \mathbf{u})$ for each interval $I \subset [0, \alpha]$ is found from (14).

Note that an iterative method of determining the elements of the orbits in the planetary many-body problem using polynomial-exponential series, and also in the form of power series with quasi-periodic coefficients, based on a system of subroutines of action with corresponding series, was developed in [7].

In conclusion we will obtain the region of the expansion of the solution of system (15) in an absolutely convergent series in powers of the initial values of the phase coordinates and the parameter λ. With the limitation that an arbitrary finite number of terms are retained in (11)–(13), the general form of the right-hand sides of Eqs (5) is expressed by the formula

$$F = \sum_{i=3}^{n} \sum_{l=1}^{N} \sum_{\varkappa, l_0, \ldots l_8} Z(s)_{\varkappa, l_0, \ldots l_8} \lambda^{\varkappa} u_1^{l_1} \ldots u_4^{l_4} w_1^{l_5} \ldots w_4^{l_8} Y_{l_0, i}(\cdot)$$

$$l = \varkappa + l_0 + l_1 + \ldots + l_8$$

Here $Z(s)_{\varkappa, l_0 \ldots l_8}$ are certain variable coefficients. The symbol $Y_{l_0, i}(\cdot)$ denotes a polynomial of degree l_0 of the argument $\cos E_i$, $\sin E_i$.

Hence, by virtue of Perron's theorem [10], for the finite interval $0 \leqslant s \leqslant \alpha$ in the region defined by the inequality

$$\frac{\lambda}{a} + \sum_{k=0}^{4} \left(\frac{|u_{0k}|}{b_k} + \frac{|w_{0k}|}{b_{k+5}} \right) \leqslant \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n(10\alpha\beta+1)} \tag{17}$$

the functions u_k and w_k, which satisfy (15), can be expanded in absolutely convergent series in powers of λ and initial values with coefficients which depend on s. Perron-type series, in view of the fact that their terms can be permuted when inequality (17) is satisfied, are identical with (16).

The constants a, b_k, b_{k+5} are found by estimating the coefficients of the expansions of the right-hand sides of (15) using the well-known rule [10]. The series on the right-hand side of inequality (17) converges by d'Alembert's test.

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